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Renormalized Kaluza-Klein theories

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Abstract: Using six-dimensional quantum electrodynamics (QED_6) as an example we study the one-loop renormalization of the theory both from the six and four-dimensional points of view. Our main conclusion is that the properly renormalized four dimensional theory never forgets its higher dimensional origin. In particular, the coefficients of the neccessary extra counterterms in the four dimensional theory are determined in a precise way. We check our results by studying the reduction of QED_4 on a two-torus.

Keywords: Field Theories in Higher Dimensions, Renormalization Regularization and Renormalons.

Contents

1. Introduction

The origin of four dimensional gauge symmetries is one of the deepest mysteries of physics. The idea of Theodor Kaluza, improved by Oskar Klein (cf. for example, [1] and references therein) that higher dimensional spacetime symmetries imply low energy gauge symmetries in four dimensions provided the extra dimensions are curled up in an appropiate way has proved quite fruitful and worth pursuing.

In the simplest setting, the Einstein-Hilbert gravitational action in a five-dimensional manifold which is a product of four-dimensional Minkowski space-time with a one-dimensional circle of radius R, looks at energies

$$E \ll M \equiv \frac{1}{R} \tag{1.1}$$

like four-dimensional Einstein-Hilbert coupled to an abelian Maxwell field.

In order to be more precise, if we believe that extra dimensions are *real*, we got to renormalize the theory. Even if we do not embed the extra-dimensional theory in some suposedly consistent framework, such as supestrings, (which would provide a cutoff of sorts), at one loop order, the fact that the higher dimensional (sometimes called the mother) theory is not renormalizable is not directly relevant, in the sense that we still can study and classify all divergences. For example, the six-dimensional electric charge is dimensionful, which allows for an unbounded number of candidate counterterms. However, to any given order in perturbation theory this number is finite, and the theory can in principle be renormalized, although it is still true that always appear new operators in the counterterms which were not present in the original lagrangian. This is then essentially a low energy approximation, because we can only expect it to be good (in the example of QED_6, in which we are going to concentrate upon) when the dimensionless quantity $\alpha_{d=6}E^2 \ll 1$, where $\alpha_{d=6}$ is the six-dimensional fine structure constant. Given the fact that the six and four-dimensional coupling constants are related by $\alpha_{d=6}M^2 \equiv \alpha_{d=4} \equiv \frac{1}{137}$, in terms of the usual four-dimensional fine-structure constant, this means $E \ll \frac{M}{\sqrt{\alpha}} \sim 10M$. It follows that one can compute reliably for energies $E \sim M$, but not much bigger. Our viewpoint will thus be that the theory is *defined* in higher dimensions by means of the necessary counterterms, in a sense that we shall try to make more precise in what follows.

At any rate, and in order to dissipate any doubts, we shall repeat in due course the same analysis on QED_4 on a two-torus. In this case the extra dimensional theory is well defined (forgetting the Landau pole), and our results are essentially the same.

Besides the six-dimensional viewpoint we are going to favor, there is always the possibility of expanding all fields in harmonics and perform the integrals over the extra dimensional compact manifold. In that way we find a four dimensional theory, but with an infinite number of fields. It seems quite intuitive that provided we keep track of the infinite set of modes, this four dimensional theory should be equivalent to the full extra-dimensional one; their respective divergences, in particular, should match. The main purpose of this paper is to check this intuition with some explicit computations. Although we are not going to work it out in any detail, it should be possible to express our results in the language of effective low energy field theories. Some steps in this direction have been already given in [2, 3].

There are then two complementary viewpoints, the higher dimensional one, and the four dimensional with the Kaluza-Klein tower, and if we want to make explicit statements on exactly when the tower begins to be relevant, we have to relate not only the classical parts, but also the quantum contributions on both sides.

Curiously enough, in the case the fields only interact through the universal coupling to an external gravitational field, the two viewpoints are *exactly* equivalent (with some qualifications).This was proved by Duff and Toms [4], and provided a strong motivation for our research.

We shall work to one loop order only. To this order, the effective action is given in terms of a functional determinant. We shall regularize it through the heat kernel approach, which respects all gauge invariances, including the geometrical ones. Let us quickly review our notation and remark on some potential ambiguities.

The geometric setting is given by a riemannian n-dimensional manifold, with a metric g_{MN}. This manifold will usually be of a factorized form: $\mathbb{R}^4 \times K$ where K is a compact $(n-4)$-dimensional manifold , and $\mathbb{R}^4$ represents the euclidean version of Minkowski space. More generally, like in the models recently popularized by Randall and Sundrum [5], this structure is present only locally, i.e., we have a fiber bundle (warped space) based on Minkowski space.

All our operators enjoy the form

$$\Delta \equiv -D_M D^M + Y \tag{1.2}$$

with

$$D_M \equiv \partial_M + X_M \tag{1.3}$$

and the operator defining the heat kernel is formally given by:

$$K(\tau) \equiv e^{-\tau\Delta} \tag{1.4}$$

acting on a convenient functional space in such a way that

$$(Kf)(x) \equiv \int \sqrt{|g|} d^n y K(x, y; \tau) f(y) \tag{1.5}$$

The short time off-diagonal [6] expansion is defined (for manifolds without boundary) by:

$$K(x, y; \tau) = K_0(x, y; \tau) \sum_{p=0} b_{2p}(x, y) \tau^p \tag{1.6}$$

where the flat space solution is given by:

$$K_0(x, y; \tau) = \frac{1}{(4\pi\tau)^{n/2}} e^{-\frac{\sigma^2}{4\tau}} \tag{1.7}$$

and σ is the geodesic distance between the two points, given in flat space by:

$$\sigma^2 = (x - y)^2 \tag{1.8}$$

and for consistency

$$b_0(x, x) = 1. \tag{1.9}$$

When boundaries are present, odd powers of $\tau^{1/2}$ do appear, which can formally be incorporated in the former expansion by allowing non vanishing odd coefficients, $b_{2p+1}\tau^{p+1/2} \neq 0$.

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It is sometimes useful to consider the integrated quantity:

$$Y(\tau, f) \equiv tr\,(f e^{-\tau\Delta}) = \sum_{k=0} \tau^{\frac{k-n}{2}} a_k(f) \tag{1.10}$$

where the trace involves whatever finite rank indices the operator might posses, and

$$a_k(f) = \frac{1}{(4\pi)^{n/2}} \int \sqrt{|g|} d^n x\, tr\, b_k(x,x) f(x) \tag{1.11}$$

The mass dimension of a_k is $k - n$, whereas the one of b_k is simply k. It follows that

$$a_0 = \frac{tr\,\mathbb{I}}{(4\pi)^{n/2}} V \equiv \frac{1}{(4\pi)^{n/2}} \int \sqrt{|g|} d^n x\, tr \mathbb{I} \tag{1.12}$$

As usual, we shall denote

$$a_k \equiv a_k(f = 1) \tag{1.13}$$

Note in particular that

$$Y(\tau) \equiv Y(\tau, f = 1) = tr\, e^{-\tau\Delta} = \sum_{k=0} \tau^{\frac{k-n}{2}} a_k \tag{1.14}$$

After all these prolegomena, the determinant is defined as:

$$\log\ \det\ \Delta = -\int_0^\infty \frac{d\tau}{\tau^{1+n/2}} \sum_{p=0} a_p \tau^{p/2} \tag{1.15}$$

There are several possible viewpoints on this integral. One of them is to analytically continue on the dimension n. The integral over the proper time τ, cut off in the infrared by $\tau_{\max} = \mu^{-2}$ produces poles in the complex variable n, given by:

$$\log\ \det\ \Delta = -\sum_{p=0} a_p \frac{2\mu^{n-p}}{p-n} + \text{finite part.} \tag{1.16}$$

which when n approaches the physical dimension, say, d,

$$n = d + \epsilon \tag{1.17}$$

yields the divergent piece of the determinant (a dimensionless quantity):

$$\log\ \det\ \Delta|_{\text{div}} = \frac{2\mu^\epsilon}{\epsilon} a_d(\Delta). \tag{1.18}$$

in the even dimensional case. This prescription yields a finite answer for odd dimensions in the absence of a boundary, and is the one usually favored when working with effective lagrangians (cf. for example [7]).

A different, and in some sense more physical possibility is to introduce a cutoff in the lower end of the proper time integral, $\Lambda/\mu \to \infty$. In that way we get, for example[1] in six dimensions:

$$\log\ \det\ \Delta|_{\text{div}} = \frac{1}{3} a_0 \Lambda^6_{(d=6)} + \frac{1}{2} a_2 \Lambda^4_{(d=6)} + a_4 \Lambda^2_{(d=6)} + a_6 \log \frac{\Lambda^2_{(d=6)}}{\mu^2_{(d=6)}} \tag{1.19}$$

[1] Although we shall try our best to avoid cluttering the notation unnecessarily, we are forced to distinguish between quantities bearing identical names, but coming from different dimensions.

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Where the heat kernel coefficients are obviously in six dimensions. In four dimensions instead:

$$\log\,\det\,\Delta|_{\rm div} = \frac{1}{2}a_0\Lambda^4_{(d=4)} + a_2\Lambda^2_{(d=4)} + a_4\log\frac{\Lambda^2_{(d=4)}}{\mu^2_{(d=4)}} \tag{1.20}$$

Where now the coefficients are the corresponding ones in four dimensions. The dominant divergence (sixth power and fourth power of the cutoff) is universal and independent of the particular operator under consideration. We shall not study it further here.

In spite of the fact that it is often pointed out that there is no way of imposing a cutoff in a gauge invariant way, we would like to stress that, at least to the one loop order, this procedure respects all gauge invariances, abelian and non abelian, as well as general covariance in its case. This is obvious, because we are *not* cutting off the momentum, but rather the *proper time*, a covariant as well as gauge invariant concept. If we remember that the proper time in the sense we are employing it, has mass dimension -2, we are neglecting in the evaluation of the one loop determinants proper times smaller than Λ^{-1}. This fact, which was probably first pointed out by Schwinger [8] in 1951, has been exploited by Bryce deWitt [6] to get covariant expansions in quantum gravity; and also by Fujikawa [9] to get the covariant anomaly.

We shall denote these two procedures *dimensional regularization* and *cutoff*, respectively. Both respect all gauge invariances of the theory but only the cutoff theory yields information on the divergences in the odd dimensional case.

2. Six dimensional quantum electrodynamics compactified on a torus

Let us now consider an example not altogether trivial, namely quantum electrodynamics (QED) on a six-dimensional manifold which is topologically four-dimensional Minkowski space times a two-torus, that is, $\mathbb{R}^4 \times S^1 \times S^1$. This example avoids the complications of interacting gravitational sectors, but in some sense is not representative of the whole Kaluza-Klein philosophy, because we are introducing gauge fields already in the extra dimensions. We are using it as a toy model.

The metric for the time being is assumed to be

$$ds^2 = \delta_{\mu\nu}dx^\mu dx^\nu + R_5^2 d\theta_5^2 + R_6^2 d\theta_6^2 \tag{2.1}$$

that is, $y_5 = R_5\theta_5$ and $y_6 = R_6\theta_6$. We shall follow consistently the convention that capital indices, like $M, N, \ldots$ run over the full dimensions, in our case from 1 to 6; greek indices, $\mu, \nu, \ldots$ run over the ordinary Minkowski coordinates, from 1 to 4; and small roman letters, $a, b, \ldots$, over the extra dimensions, that is, from 5 to 6.

The (euclidean version of the) action then reads

$$S = \int d^6x \left(\frac{1}{4}F^2_{MN} + \bar{\psi}(\not{D} + m)\psi\right) \tag{2.2}$$

where the abelian covariant derivative is simply:

$$D_M\psi \equiv (\partial_M - eA_M)\,\psi \tag{2.3}$$

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Let us recall here that, for vanishing curvature, the general formulas [10] for the first few coefficients of an operator of the form (1.2) are:

$$
\begin{aligned}
a_2 &= -\int \frac{d^n x}{(4\pi)^{\frac{n}{2}}}\, tr\, Y && (2.4)\\
a_4 &= \int \frac{d^n x}{(4\pi)^{\frac{n}{2}}}\, tr\left(\frac{1}{12}X_{MN}^2 + \frac{1}{2}Y^2 - \frac{1}{6}Y_{;MM}\right) && (2.5)\\
a_6 &= \frac{1}{360}\int \frac{d^n x}{(4\pi)^{\frac{n}{2}}}\, tr\Big(8X_{MN;R}^2 + 2X_{MN;N}^2 + \\
&\quad + 12X_{MN;RR}X^{MN} - 12X_{MN}X^{NR}X_R{}^M - 6Y_{;MMNN} + \\
&\quad + 60YY_{;MM} + 30Y_{;M}^2 - 60Y^3 - 30YX_{MN}^2\Big) && (2.6)
\end{aligned}
$$

Where ; denotes covariant derivative, and

$$X_{MN} = \partial_M X_N - \partial_N X_M + [X_M, X_N] \tag{2.7}$$

In order to perform the explicit computation, it is exceedingly useful to combine the fermionic and bosonic sectors in a full supermatrix. Please read the appendix for a brief review of the technique and notation.

Computing the coefficients is then straightforward albeit somewhat laborious. In terms of the background fields $\bar{A}_M, \eta, \bar{\eta}$

$$a_2 = \int \frac{d^6 x}{(4\pi)^3} 8m^2 \tag{2.8}$$

as well as

$$a_4 = \int \frac{d^6 x}{(4\pi)^3}\left(\frac{4}{3}e^2\bar{F}_{MN}^2 + 4e^2\bar{\eta}\bar{\not{D}}\eta + 12me^2\bar{\eta}\eta\right) \tag{2.9}$$

Finally we get (using the background equations of motion):

$$
\begin{aligned}
a_6 = \int \frac{d^6 x}{(4\pi)^3}\Big(&-\frac{1}{12}e^4\bar{\eta}\Sigma_{MNL}\eta\bar{\eta}\Sigma^{MNL}\eta + \frac{19}{15}e^2 m\bar{\eta}\bar{D}_M\bar{D}^M\eta + \frac{2}{15}e^3\bar{\eta}\gamma_N\bar{D}_M\eta\bar{F}^{MN} - \\
&- e^3 m\bar{\eta}\gamma_M\gamma_N\eta\bar{F}_{MN} - 2e^2m^2\bar{\eta}\gamma^M\bar{D}_M\eta - 6e^2m^3\bar{\eta}\eta - \frac{11}{45}e^2\bar{D}_R\bar{F}_{MN}\bar{D}^R\bar{F}^{MN} + \\
&+ \frac{23}{9}e^2\bar{D}_M\bar{F}^{MN}\bar{D}^R\bar{F}_{RN} - \frac{4}{3}e^2m^2\bar{F}_{MN}\bar{F}^{MN}\Big)
\end{aligned} \tag{2.10}
$$

Where Σ_{MNL} is the totally antisymmmetric product of three gammas. Remember that in dimensional regularization

$$\Delta S = \frac{1}{\epsilon}a_6 \tag{2.11}$$

plus a possible finite part. With a cutoff, these are the logarithmic divergences, and we have in addition both quadratic and quartic divergences, on which more to follow.

The first conclusion we can draw from this analysis is that quantum effects, besides renormalizing the six-dimensional couplings, induce a set of non-minimal interactions which are generated with arbitrary coefficients.

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Actually, due to the fact that the mass dimension of the coupling constant is -1, there is no finite closed set of operators of counterterms. Let us be more specific.

First of all, there is a dimension five operator, which becomes a potential counterterm in the massive case:

$$\mathcal{O}_{(5)} = \left(\bar{\psi}\psi\right) \tag{2.12}$$

The set of gauge-invariant dimension six operators is given by:

$$\mathcal{O}^i_{(6)} = \left(\bar{\psi}\not{D}\psi,\, F^2_{MN}\right) \tag{2.13}$$

To the next order, that is, dimension seven, the list reads:

$$\mathcal{O}^i_{(7)} = \left(\bar{\psi}\not{D}\not{D}\psi\right) \tag{2.14}$$

The dimension eight operators are:

$$\mathcal{O}^i_{(8)} = \left(\bar{\psi}\not{D}\not{D}\not{D}\psi,\, \bar{\psi}\sigma_{MN}\psi F^{MN},\, D^M F_{MN} D_R F^{RN},\, F_{NL}D^2 F^{NL}\right) \tag{2.15}$$

And finally, to dimension nine we have to consider:

$$\mathcal{O}^i_{(9)} = \left(\bar{\psi}\gamma_M D_N \psi F^{MN},\, \bar{\psi}D_A D_B D_C D_D \psi t^{ABCD}\right) \tag{2.16}$$

In the massive case the dimension of this operators can be increased by introducing powers of m. Amongst the operators that actually appear as counterterms only the $\mathcal{O}^2_{(8)}$ is absent. At any rate it should be plain that we can claim results only to first nontrivial order in the six-dimensional fine structure constant, and that we have really no right to keep the e^3 and e^4 terms in the counterterm.

The non renormalizabity of the theory manifests itself in the fact that if we were to include all those dimension seven and dimension eight operators, they would generate more and more higher dimension operators as counterterms. There is no closed set, unless we assume, as is natural to the order we are working, that the effect of all those couplings is of higher order in the six-dimensional fine structure constant.[2]

[2] Keeping in mind that we are not performing a fully consistent computation, if we define the renormalization constants as is usually done:

$$\begin{aligned} A_0 &= Z_3^{1/2} A \\ \psi_0 &= Z_2^{1/2}\psi \\ e_0 &= Z_1 Z_2^{-1} Z_3^{-1/2} e \\ m_0 &= Z_m m \end{aligned} \tag{2.17}$$

we easily get $Z_1 = Z_2$ which conveys the fact that the theory is gauge invariant, and

$$\begin{aligned} Z_2 &= 1 - \frac{e^2 m^2}{32\pi^3\epsilon} \\ Z_3 &= 1 - \frac{e^2 m^2}{12\pi^3\epsilon} \\ Z_m &= 1 - \frac{e^2 m^2}{16\pi^3\epsilon}\,. \end{aligned} \tag{2.18}$$

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3. The four-dimensional viewpoint

Let's consider the point of view of the reduced theory. We can now expand all fields in Fourier series:

$$\phi(x,y) = \frac{1}{2\pi\sqrt{R_5 R_6}} \sum_n \phi_n(x) e^{i\frac{n}{R}\cdot y} \tag{3.1}$$

where $n \equiv (n_5, n_6)$, and we have included a convenient factor in front to take care of the diference of canonical dimensions of the fields in six and four dimensions. Real fields (such as the photon) obey

$$\phi_n^*(x) = \phi_{-n}(x) \tag{3.2}$$

The six-dimensional gamma matrices can be chosen such as:

$$\begin{aligned} \gamma_\mu^{(6)} &= \sigma_3 \otimes \gamma_\mu^{(4)} \\ \gamma_5^{(6)} &= \sigma_1 \otimes 1 \\ \gamma_6^{(6)} &= \sigma_2 \otimes 1 \end{aligned} \tag{3.3}$$

In that way, six-dimensional spinors split in two four-dimensional ones:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{3.4}$$

A simple calculation then leads to the renormalization group functions:

$$\begin{aligned} \beta_e &\equiv \frac{\partial e}{\partial \log \mu} = -\frac{1}{24\pi^3} e^3 m^2 \\ \beta_m &\equiv \frac{\partial m}{\partial \log \mu} = \frac{1}{16\pi^3} e^2 m^3 \end{aligned} \tag{2.19}$$

The renormalization of the fermion mass is entangled with the charge renormalization. The behavior of the coupling constants reads:

$$\begin{aligned} e &= e_0 - \frac{1}{24\pi^3} m_0^2 e_0^3 \log \mu/\mu_0 \\ m &= m_0 \left(1 - \frac{1}{24\pi^3} m_0^2 e_0^2 \log \mu/\mu_0\right)^{-3/2} \end{aligned} \tag{2.20}$$

The dimensionful charge vanishes when

$$\mu = \mu_0 e^{\frac{24\pi^3}{m_0^2 e_0^2}} \tag{2.21}$$

If we define the dimensionless couplings

$$\begin{aligned} \hat{e} &\equiv e\mu \\ \hat{m} &\equiv \frac{m}{\mu} \end{aligned} \tag{2.22}$$

then the renormalization group equations read

$$\begin{aligned} \beta_{\hat{e}} &= \hat{e} - \frac{1}{24\pi^3} \hat{m}^2 \hat{e}^3 \\ \beta_{\hat{m}} &= -\hat{m} + \frac{1}{16\pi^3} \hat{e}^2 \hat{m}^3 . \end{aligned} \tag{2.23}$$

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It is a simple matter to perform the integrals over the angular variables and obtain the gauge fixed action (still exact) in the four dimensional form:

$$S = \int d^4x \sum_{n_5,n_6} \left(\bar{\psi}^1_n \partial\!\!\!/ \psi^1_n + \bar{\psi}^2_n \partial\!\!\!/ \psi^2_n + \bar{\psi}^1_n \left(i\frac{n_5}{R_5} + \frac{n_6}{R_6} \right) \psi^2_n - \bar{\psi}^2_n \left(i\frac{n_5}{R_5} - \frac{n_6}{R_6} \right) \psi^1_n + \right.$$
$$+ m \left(\bar{\psi}^1_n \psi^1_n - \bar{\psi}^2_n \psi^2_n \right) - \frac{1}{2}(A^n_\mu)^* \left(\Box - \frac{n_5^2}{R_5^2} - \frac{n_6^2}{R_6^2} \right) A^\mu_n - \frac{1}{2}(A^n_5)^* \left(\Box - \frac{n_5^2}{R_5^2} - \frac{n_6^2}{R_6^2} \right) A^n_5 -$$
$$- \frac{1}{2}(A^n_6)^* \left(\Box - \frac{n_5^2}{R_5^2} - \frac{n_6^2}{R_6^2} \right) A^n_6 - e \sum_m \left(\bar{\psi}^1_m A\!\!\!/_{m-n} \psi^1_n + \bar{\psi}^2_m A\!\!\!/_{m-n} \psi^2_n + \right.$$
$$\left. \left. + \bar{\psi}^1_m A_5^{m-n} \psi^2_n - \bar{\psi}^2_m A_5^{m-n} \psi^1_n - i\bar{\psi}^1_m A_6^{m-n} \psi^2_n - i\bar{\psi}^2_m A_6^{m-n} \psi^1_n \right) \right) \tag{3.5}$$

and the four-dimensional coupling constant is

$$e \equiv \frac{e^{(6)}}{2\pi\sqrt{R_5 R_6}} \equiv e^{(6)} M \,. \tag{3.6}$$

Here we see clearly a generic feature of interacting theories, namely that there is no consistent truncation in the sense that all massive fields interact among themselves and with the massless fields.

3.1 Gauge symmetries of the four-dimensional action

Six-dimensional QED has an obvious U(1) symmetry. It is interesting to see how this invariance is traduced in the lower dimensional theory. Before gauge fixing, the four-dimensional action enjoys the infinite set of symmetries:

$$\begin{aligned} \delta A^n_\mu &= i\partial_\mu \Lambda_n \\ \delta A^n_5 &= -\frac{n_5}{R_5}\Lambda_n \\ \delta A^n_6 &= -\frac{n_6}{R_6}\Lambda_n \end{aligned} \tag{3.7}$$

Where Λ_n are the modes of the expansion of the abelian transformation parameter. All those gauge symmetries Λ_{n_5,n_6} are spontaneously broken, except for the zero mode, corresponding to $\Lambda_{0,0}$. The A^n_μ are the massive vector bosons, and the A^n_5 and A^n_6 the scalar higgses.

There is a curious fact, however, and this is the appearance of two singlets in four dimensions, namely A^0_5 and A^0_6. Those singlets are massless at tree level, but no symmetry protects them from getting massive through quantum corrections.

The same fields are protected from getting masses in six dimensions, through gauge invariance and six dimensional Lorentz covariance. The point is that the breaking

$$O(1,5) \to O(1,3) \times O(2) \times O(2) \tag{3.8}$$

of the symmetry group of the vacuum is an instance of spontaneous compactification; i.e., the equations of motion enjoy the full $O(1,5)$ symmetry, and only the solution breaks it.

3.2 The massless action

The zero mode of the above action is

$$S_{zm} = \int d^4x \left(\bar{\psi}^1 \not{\partial}\psi^1 + \bar{\psi}^2 \not{\partial}\psi^2 + m\left(\bar{\psi}^1\psi^1 - \bar{\psi}^2\psi^2\right) - \frac{1}{2}A_\mu \Box A^\mu - \right.$$
$$\left. -\frac{1}{2}\phi^* \Box \phi - e\left(\bar{\psi}^1 \not{A}\psi^1 + \bar{\psi}^2 \not{A}\psi^2 + \bar{\psi}^1 \phi \psi^2 - \bar{\psi}^2 \phi^* \psi^1\right)\right) \tag{3.9}$$

where we have represented the zero modes of all fields by the same letter without any subindex:

$$A_5^0 - iA_6^0 \equiv \phi^0 \equiv \phi \tag{3.10}$$

It must be stressed that this is *not* a consistent truncation,(in the sense of the word usually employed in supergravity and superstrings) owing to the fact that both A_μ^0 and ϕ couple diagonally to the whole fermionic tower; it is expected, however, to be a physically sensible one at energies $E \ll M$.

Denoting $\bar{\phi}$ the background for ϕ the cuadratic part of the action is

$$S_{zm} = \int d^4x \left(\bar{\psi}^1 \not{\partial}\psi^1 + \bar{\psi}^2 \not{\partial}\psi^2 + m\left(\bar{\psi}^1\psi^1 - \bar{\psi}^2\psi^2\right) - \frac{1}{2}\phi_\mu \Box \phi^\mu - \right.$$
$$-\frac{1}{2}\phi^* \Box \phi - e\left(\bar{\psi}^1 \bar{\not{A}}\psi^1 + \bar{\psi}^2 \bar{\not{A}}\psi^2 + \bar{\eta}^1 \gamma^\mu \phi_\mu \psi^1 + \bar{\eta}^2 \gamma^\mu \phi_\mu \psi^2 + \bar{\psi}^1 \gamma^\mu \phi_\mu \eta^1 + \right.$$
$$\left.\left. + \bar{\psi}^2 \gamma^\mu \phi_\mu \eta^2 + \bar{\psi}^1 \bar{\phi}\psi^2 - \bar{\psi}^2 \bar{\phi}^* \psi^1 + \bar{\eta}^1 \phi \psi^2 - \bar{\eta}^2 \phi^* \psi^1 + \bar{\psi}^1 \phi \eta^2 - \bar{\psi}^2 \phi^* \eta^1\right)\right) \tag{3.11}$$

Where e is now the four dimensional coupling. The first two coefficients in the heat kernel expansion are:

$$a_2^{(zm)} = \int \frac{d^4x}{(4\pi)^2} 8\left(m^2 - e^2|\bar{\phi}|^2\right) \tag{3.12}$$

and

$$a_4^{(zm)} = \int \frac{d^4x}{(4\pi)^2} \left(\frac{4}{3}e^2 \bar{F}^2_{\mu\nu} - 4e^2 \bar{\phi}^* \Box \bar{\phi} + 8e^2 m^2 |\bar{\phi}|^2 - 4e^4 |\bar{\phi}|^4 + \right.$$
$$\left. + 4e^2\left(\bar{\eta}^1 \bar{\not{D}}\eta^1 + \bar{\eta}^2 \bar{\not{D}}\eta^2\right) + 12me^2\left(\bar{\eta}^1\eta^1 - \bar{\eta}^2\eta^2\right) + 8e^3 \bar{\eta}^2 \bar{\phi}^* \eta^1 - 8e^3 \bar{\eta}^1 \bar{\phi}\eta^2\right) \tag{3.13}$$

This is the logarithmically divergent counterterm that arises when renormalizing the zero mode of the four dimensional action.

It should be remarked that the resulting four dimensional model is superficially very similar to the Coleman-Weinberg setup, in which radiative spontaneous symmetry breaking was first discovered. There is a crucial difference though, and this is that the scalar field is not charged, in spite of being complex. The reason is that it remembers its gauge origin, and six-dimensional gauge invariance manifests here as a Kac-Moody transformation acting on the full tower of massive states. In addition to that, the quartic coupling is here a quantum effect, because it was not present in the bare four-dimensional lagrangian. Also the scalar field gets massive, with a mass proportional to the fermion mass (times the four-dimensional fine structure constant).[3]

[3] At any rate, this yields (twice) the usual beta function for the four dimensional fine structure constant:

$$\beta_e = \frac{1}{6\pi^2}e^3 \tag{3.14}$$

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4. A comparison of the massless sector of the full six-dimensional divergences with the divergences of the massless sector of the four-dimensional theory

After all this work, we are finally in a position to study our main concern, namely, how the divergent part of the six-dimensional effective action is related to the corresponding four-dimensional quantity.

4.1 The cutoff theory

Let us first analyze the problem from the viewpoint of the cutoff theory. As we have seen in six dimensions the divergent part of the effective action is given through the equation (1.19); while from the four-dimensional viewpoint the corresponding formula stems from (1.20).

When we are interested in the zero mode, i.e., the piece in six dimensions where all fields are independent of the extra dimensions, the measure clearly factorizes:

$$d^6x \to \frac{1}{M^2} d^4x \tag{4.1}$$

It is plain that the divergences never coincide exactly. The *only* way to make the divergences related to the fourth heat-kernel coefficient identical in six and in four dimensions is choose different proper time cutoffs in both dimensions in such a way that:

$$\frac{\Lambda^2_{(d=6)}}{M^2} \equiv \log \frac{\Lambda^2_{(d=4)}}{\mu^2_{(d=4)}} \tag{4.2}$$

We choose that because those coefficients are *almost* identical, so that the logarithmic divergences are as similar as possible.

This identification leads to the reinterpretation of the six-dimensional quartic divergences as $\log^2$:

$$\Lambda^4_{(d=6)} \to M^4 \left(\log \frac{\Lambda^2_{(d=4)}}{\mu^2_{(d=4)}} \right)^2 \tag{4.3}$$

and finally, the six-dimensional logarithmic divergences appear in the guise of $\log \log$.

$$\log \frac{\Lambda^2_{(d=6)}}{\mu^2_{(d=6)}} \to \log \left(\frac{M^2}{\mu^2_{(d=6)}} \log \frac{\Lambda^2_{(d=4)}}{\mu^2_{(d=4)}} \right) \tag{4.4}$$

This reinterpretation gives rise to a few more four-dimensional nonstandard counterterms, which we will comment upon in a moment.

The behavior of the charge is:

$$e^2 = \frac{e_0^2}{1 - \frac{e_0^2}{3\pi^2} \log \mu/\mu_0} \tag{3.15}$$

which blows up at a Landau pole located at

$$\Lambda \equiv \mu_0 e^{3\pi^2/e_0^2} \tag{3.16}$$

Let us stress, for the time being, that the logarithmic divergence, when renormalizing (correctly) from six dimensions is not identical to the one (3.13), but rather

$$\begin{aligned}\Delta S_{\log} \equiv \int \frac{d^4x}{(4\pi)^3} e^2 \Big(& 4\left(\bar{\eta}^1 \not{\partial} \eta^1 + \bar{\eta}^2 \not{\partial} \eta^2\right) + \\ & + 12m\left(\bar{\eta}^1\eta^1 - \bar{\eta}^2\eta^2\right) + \frac{4}{3}\left(\bar{F}^{\mu\nu}\bar{F}_{\mu\nu} - \right. \\ & \left. - 2\bar{A}_5 \Box \bar{A}_5 - 2\bar{A}_6 \Box \bar{A}_6\right) - \\ & - 4e\left(\bar{\eta}^1 \bar{\not{A}} \eta^1 + \bar{\eta}^2 \bar{\not{A}} \eta^2 + \bar{\eta}^1 \bar{\phi} \eta^2 - \bar{\eta}^2 \bar{\phi}^* \eta^1\right) \Big) \log \frac{\Lambda^2_{d=4}}{\mu^2_{d=4}} \end{aligned} \tag{4.5}$$

The scalars A_5 and A_6 are now protected by the six dimensional symmetries, as they should be.

4.2 Dimensional regularization

Were we to stick to dimensional regularization, we would have to compare the four dimensional counterterm with the massless sector of the six-dimensional one, which was previously determined in equation (2.10). There are then two types of terms.

First of all, those terms which have negative dimension constants in front, which are precisely the ones not present in the original six-dimensional lagrangian, yield in four dimensions counterterms with dimension six operators, suppressed by two powers of the Kaluza-Klein scale:

$$\begin{aligned}\Delta S_{(1)} = \frac{e^2}{64\pi^3 M^2 \epsilon} \int d^4x \Big(& -\frac{1}{12} e^2 \left(\bar{\eta} \Sigma_{\mu\nu\rho} \eta\right)^2 + \frac{19}{15} m \bar{\eta} \bar{D}_\mu \bar{D}^\mu \eta + \\ & + \frac{2}{15} e \bar{\eta} \gamma_\nu \bar{D}_\mu \eta \bar{F}^{\mu\nu} - e m \bar{\eta} \gamma^\mu \gamma^\nu \eta \bar{F}_{\mu\nu} - \frac{11}{45}\left(\bar{D}_\lambda \bar{F}_{\mu\nu}\right)^2 + \frac{23}{9} (\bar{D}_\mu \bar{F}_{\mu\nu})^2 + \\ & + \cdots \Big) \end{aligned} \tag{4.6}$$

Where the dots stand for terms with contractions of index in the extra dimensions and e is the four-dimensional coupling. Then, there are the usual four-dimensional counterterms in the guise

$$\Delta S_{(2)} = -\frac{2e^2 m^2}{64\pi^3 M^2 \epsilon} \int d^4x \left(\bar{\eta} \bar{\not{D}} \eta + 3m\bar{\eta}\eta + \frac{2}{3}\bar{F}^2_{\mu\nu}\right) \tag{4.7}$$

The six-dimensional mass m^2 can clearly be tuned so as to survive in the limit in which the Kaluza-Klein scale is pushed to infinity. We simply have to tune the dimensionless quantity

$$\frac{e^2 m^2}{64\pi^3 M^2 \epsilon} \tag{4.8}$$

towards the true four-dimensional $\frac{e^2}{16\pi^2\epsilon}$, while keeping the six-dimensional mass m in their four-dimensional value. In such a way we recover *almost* all four dimensional counterterms, albeit with a different sign, which could be accounted for by changing the direction of the analytical continuation: $\epsilon_{d=6} = -\epsilon_{d=4}$.

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We say almost, because it can easily be seen from these results that there is no room for the $|\phi|^2$ and $|\phi|^4$ counterterms, which appear when working upwards from four-dimensions, but do not appear in the zero mode of the six-dimensional counterterm.

The only (dim) hope is that these four-dimensional counterterms are actually cancelled when the full tower of Kaluza-Klein states is considered. The next subsection is devoted to disipate this lingering doubt.

It seems indeed strange that no quartic interaction is generated when coming from six dimensions. No definite conclusions can be draw, however, because those effects are of order $O(\lambda^2)$, where λ is que quartic coupling constant, which means order $O(e^8)$ in our case. We have no right to keep those terms.

There is a very simple mapping from six-dimensional operators to four-dimensional ones, namely

$$\mathcal{O}_{(n)} \to \mathcal{O}_{(n-N)} \tag{4.9}$$

where N is the number of fields involved in the operator.

In that way it is seen that the reduction works at follows:

$$\begin{aligned} \mathcal{O}_{(5)} &\to \mathcal{O}_{(3)} \\ \mathcal{O}_{(6)} &\to \mathcal{O}_{(4)} \\ \mathcal{O}_{(7)} &\to \mathcal{O}_{(5)} \\ \mathcal{O}_{(8)} &\to \mathcal{O}_{(6)} \end{aligned} \tag{4.10}$$

except for

$$\mathcal{O}^2_{(8)} \to \mathcal{O}^2_{(5)} \tag{4.11}$$

In four dimensions, all operators with dimension higher than four appear necessarily with coefficients which get inverse powers of the compactification scale, M. We should be then pretty confident of all results gotten in the limit in which this scale goes to infinity.

Another question is what happens in the chiral limit. If the mass of the fermion vanishes, then the six-dimensional counterterms do *not* include the four-dimensional ones. If we think about it, the conclusion is almost unavoidable, because there is no parameter in the lagrangian with the dimension of mass. The inverse coupling constant does not qualify for this, because it is never going to appear in a perturbative computation.

5. The full tower of four-dimensional divergences

Let us consider now the problem of the divergences of the four-dimensional theory with the whole Kaluza-Klein tower. We intend to compute the counterterm asociated with the full four-dimensional Lagrangian (3.5). We let the index $n = (n_5, n_6)$ run over the whole tower of each field. Notice that the bosonic fields are now complex (except the one corresponding to $n = 0$). N is the complex mass number $N = \frac{n_6}{R_6} + i\frac{n_5}{R_5}$, and also $L = \frac{l_6}{R_6} + i\frac{l_5}{R_5}$, We have also defined $\bar{\phi}_n \equiv \bar{A}^n_5 - i\bar{A}^n_6$ and $\bar{\phi}^*_n \equiv \bar{A}^n_5 + i\bar{A}^n_6 \neq (\bar{\phi}_n)^* = \bar{A}^{-n}_5 + i\bar{A}^{-n}_6$.

As we have said the massive ($n \neq 0$) modes are complex. In order to use the algorithm explained in the appendix we have to double this modes into real and imaginary parts. However it is also possible to do the calculatios with the complex fields and introduce at the end some extra factors in the adecuate terms. After squaring the matrices and performing the supertraces we get the following counterterms in four dimensions with some labor

$$a_2 = \int \frac{d^4x}{(4\pi)^2} \sum_l \left(8m^2 - 4|L|^2 - 8e^2 \sum_n \bar{\phi}^*_n \bar{\phi}_{-n} + 8e \left(L^* \bar{\phi}_0 - L \bar{\phi}^*_0 \right) \right) \tag{5.1}$$

The mode sum can be regularized and performed with the help of a zeta function. We shall do it in the next section, when working out the reduction of QED_4 on a two-torus. The fourth heat-kernel coefficient is[4] quite messy indeed. At least, one thing is clear: there is no way to perform a clever resummation (like the one Duff and Toms did in the free case) in order to cancel the four dimensional counterterms for both $|\phi|^2$ and $|\phi|^4$, for the simple reason that there is no contribution of the massive fields to them. This fact was not obvious *a priori* and the doubt about it was the main reason why this computation was performed.

6. The true four-dimensional renormalization

From our point of view, in which the full theory is defined in six dimensions, the true renormalization is the one that is obtained via an harmonic expansion of the six-dimensional counterterm(s).

---

[4] Here is the explicit expression

$$\begin{aligned}
a_4 = \int \frac{d^4x}{(4\pi)^2} \sum_l \Bigg(& \left(-4m^4 + 2|L|^4 - 8m^2|L|^2\right) + \frac{4}{3} e^2 \sum_n \bar{F}^n_{\mu\nu} \bar{F}^{\mu\nu}_{-n} - 4e^2 \sum_n \bar{\phi}^*_n \Box \bar{\phi}_{-n} - \\
& -8e \left(m^2 + |L|^2\right) \left(L \bar{\phi}^*_0 - L^* \bar{\phi}_0\right) + 8e^2 \sum_n \left(m^2 + |L+N|^2 + |N|^2\right) \bar{\phi}^*_n \bar{\phi}_{-n} - \\
& -4e^2 \sum_n \left(N^* + L^*\right) L^* \bar{\phi}_n \bar{\phi}_{-n} - 4e^2 \sum_n \left(N + L\right) L \bar{\phi}^*_n \bar{\phi}^*_{-n} + \\
& + 8e^3 \sum_{m,n} \bar{\phi}^*_{m-l} \bar{\phi}_{l-n} \left(M \bar{\phi}^*_{n-m} - N^* \bar{\phi}_{n-m}\right) + 4e^2 \sum_{m,n,s} \bar{\phi}^*_{m-l} \bar{\phi}_{l-s} \bar{\phi}^*_{s-n} \bar{\phi}_{n-m} - \\
& -4e^2 \sum_n N^* \partial_\mu \bar{\phi}_n \bar{A}^\mu_{-n} + 4e^2 \sum_n N \partial_\mu \bar{\phi}^*_n \bar{A}^\mu_{-n} + 4e^2 \sum_n |N|^2 \bar{A}^n_\mu \bar{A}^\mu_{-n} + \\
& +8e^2 \sum_{n \neq 0} \left(\bar{\eta}^1_{l-n} \not{\partial} \eta^1_{l-n} + \bar{\eta}^2_{l-n} \not{\partial} \eta^2_{l-n}\right) - 8e^3 \sum_{m \neq 0,n} \left(\bar{\eta}^1_{l-m} \not{\bar{A}}_{l-n} \eta^1_{n-m} + \bar{\eta}^2_{l-m} \not{\bar{A}}_{l-n} \eta^2_{n-m}\right) + \\
& +24me^2 \sum_{n \neq 0} \left(\bar{\eta}^1_{l-n} \eta^1_{l-n} - \bar{\eta}^2_{l-n} \eta^2_{l-n}\right) + 16e^3 \sum_{m \neq 0,n} \bar{\eta}^2_{l-m} \bar{\phi}^*_{l-n} \eta^1_{n-m} - 16e^3 \sum_{m \neq 0,n} \bar{\eta}^1_{l-m} \bar{\phi}_{l-n} \eta^2_{n-m} + \\
& +16e^2 \sum_{n \neq 0} L^* \bar{\eta}^2_{l-n} \eta^1_{l-n} + 16e^2 \sum_{n \neq 0} L \bar{\eta}^1_{l-n} \eta^2_{l-n} + 4e^2 \left(\bar{\eta}^1_l \not{\partial} \eta^1_l + \bar{\eta}^2_l \not{\partial} \eta^2_l\right) - \\
& -4e^3 \sum_n \left(\bar{\eta}^1_n \not{\bar{A}}_{n-l} \eta^1_l + \bar{\eta}^2_n \not{\bar{A}}_{n-l} \eta^2_l\right) + 12me^2 \left(\bar{\eta}^1_l \eta^1_l - \bar{\eta}^2_l \eta^2_l\right) + 8e^3 \sum_n \bar{\eta}^2_n \bar{\phi}^*_{n-l} \eta^1_l - \\
& -8e^3 \sum_n \bar{\eta}^1_n \bar{\phi}_{n-l} \eta^2_l + 8e^2 L^* \bar{\eta}^2_l \eta^1_l + 8e^2 L \bar{\eta}^1_l \eta^2_l \Bigg)
\end{aligned} \tag{5.2}$$

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6.1 The cutoff theory

With the interpretation of the six-dimensional cutoff we have advocated, the four-dimensional logarithmic divergences read

$$\begin{aligned}\Delta S_{\log} \equiv \int \frac{d^4x}{(4\pi)^3} e^2 \sum_n \Bigg(& 4\left(\bar\eta^1_n \partial\!\!\!/ \eta^1_n + \bar\eta^2_n \partial\!\!\!/ \eta^2_n + N\bar\eta^1_n\eta^2_n + N^*\bar\eta^2_n\eta^1_n\right) + 12m\left(\bar\eta^1_n\eta^1_n - \bar\eta^2_n\eta^2_n\right) + \\ & + \frac{4}{3}\left(\bar F^{\mu\nu}_{-n}\bar F^n_{\mu\nu} + 2|N|^2 \bar A^\mu_{-n}\bar A^n_\mu - 4i\partial_\mu \bar A^\mu_{-n}\left(\frac{n_5}{R_5}\bar A^n_5 + \frac{n_6}{R_6}\bar A^n_6\right) + \right. \\ & \left. + 2\bar A_5^{-n}\left(-\Box + \frac{n_6^2}{R_6^2}\right)\bar A^n_5 + 2\bar A_6^{-n}\left(-\Box + \frac{n_5^2}{R_5^2}\right)\bar A^n_6 - 4\frac{n_5 n_6}{R_5 R_6}\bar A_5^{-n}\bar A^n_6\right) - \\ & - 4e\sum_m \left(\bar\eta^1_m \bar A\!\!\!/_{m-n}\eta^1_n + \bar\eta^2_m \bar A\!\!\!/_{m-n}\eta^2_n + \bar\eta^1_m\bar\phi_{m-n}\eta^2_n - \bar\eta^2_m\bar\phi^*_{m-n}\eta^1_n\right)\Bigg)\log\frac{\Lambda^2_{d=4}}{\mu^2_{d=4}} \end{aligned} \tag{6.1}$$

In addition to that, there are the $\log^2$ divergences, coming from the quartic divergences in six dimensions. Those are trivial in our case, because they do not depend on the background fields.

Finally, there are the log log divergences, stemming from the logarithmic divergence in six dimensions. This divergence is suppressed by the scale of compactification. The result of a somewhat heavy computation, keeping terms up to cubic order in the four-dimensional electric charge, is:

$$\begin{aligned}\Delta S_{\log\log} \equiv \int \frac{d^4x}{(4\pi)^3}\frac{e^2}{M^2}\sum_n \Bigg(& -m\bar\eta^1_n\left(\frac{19}{15}\left(-\Box + |N|^2\right) + 2m\partial\!\!\!/ + 6m^2\right)\eta^1_n + \\ & + m\bar\eta^2_n\left(\frac{19}{15}\left(-\Box + |N|^2\right) - 2m\partial\!\!\!/ + 6m^2\right)\eta^2_n - 2m^2\left(N\bar\eta^1_n\eta^2_n + N^*\bar\eta^2_n\eta^1_n\right) + \\ & + \frac{23}{9}\partial_\mu \bar F^{\mu\nu}_{-n}\left(\partial^\rho \bar F^n_{\rho\nu} - 2|N|^2\bar A^n_\nu\right) - \bar F^{-n}_{\mu\nu}\left(\frac{11}{45}\left(-\Box + |N|^2\right) + \frac{4}{3}m^2\right)\bar F^{\mu\nu}_n + \\ & + |N|^2 \bar A^\mu_{-n}\left(\frac{31}{15}\left(-\Box + |N|^2\right) - \frac{8}{3}m^2\right)\bar A^n_\mu - \\ & - i\partial_\mu \bar A^\mu_{-n}\left(\frac{62}{15}\left(-\Box + |N|^2\right) - \frac{16}{3}m^2\right)\left(\frac{n_5}{R_5}\bar A^n_5 + \frac{n_6}{R_6}\bar A^n_6\right) + \\ & + \bar A_5^{-n}\left(\frac{31}{15}\left(-\Box + |N|^2\right) - \frac{8}{3}m^2\right)\left(-\Box + \frac{n_6^2}{R_6^2}\right)\bar A^n_5 + \\ & + \bar A_6^{-n}\left(\frac{31}{15}\left(-\Box + |N|^2\right) - \frac{8}{3}m^2\right)\left(-\Box + \frac{n_5^2}{R_5^2}\right)\bar A^n_6 - \\ & - \frac{n_5 n_6}{R_5 R_6}\bar A_5^{-n}\left(\frac{62}{15}\left(-\Box + |N|^2\right) - \frac{16}{3}m^2\right)\bar A^n_6\Bigg)\log\left(\frac{M^2}{\mu^2_{(d=6)}}\log\frac{\Lambda^2_{(d=4)}}{\mu^2_{(d=4)}}\right) + \\ & + O(e^3) \end{aligned} \tag{6.2}$$

6.2 Dimensional regularization

In that case the true divergences only come from the sixth coefficient, which yields the log log divergences we just wrote down.

This means that in addition to the already mentioned counterterms to the zero modes there are a full tower of counterterms involving six-dimensional operators.

It is of interest to specialize to the massless case ($m = 0$), in which, as we have already noticed, no ordinary dimension four operator is recovered:

$$
\begin{aligned}
a_6 = & \int \frac{d^4x}{(4\pi)^3}\frac{e^2}{M^2}\sum_n \left(\frac{23}{9}\partial_\mu \bar{F}^{\mu\nu}_{-n}\left(\partial^\rho \bar{F}^n_{\rho\nu} - 2|N|^2\bar{A}^n_\nu\right) - \frac{11}{45}\bar{F}^{-n}_{\mu\nu}\left(-\Box + |N|^2\right)\bar{F}^{\mu\nu}_n + \right.\\
& + \frac{31}{15}|N|^2\bar{A}^\mu_{-n}\left(-\Box + |N|^2\right)\bar{A}^n_\mu - i\frac{62}{15}\partial_\mu \bar{A}^\mu_{-n}\left(-\Box + |N|^2\right)\left(\frac{n_5}{R_5}\bar{A}^n_5 + \frac{n_6}{R_6}\bar{A}^n_6\right) + \\
& + \frac{31}{15}\bar{A}_5^{-n}\left(-\Box + |N|^2\right)\left(-\Box + \frac{n_6^2}{R_6^2}\right)\bar{A}^n_5 + \frac{31}{15}\bar{A}_6^{-n}\left(-\Box + |N|^2\right)\left(-\Box + \frac{n_5^2}{R_5^2}\right)\bar{A}^n_6 - \\
& \left. - \frac{62}{15}\frac{n_5 n_6}{R_5 R_6}\bar{A}_5^{-n}\left(-\Box + |N|^2\right)\bar{A}^n_6\right) + O(e^3)
\end{aligned}
\tag{6.3}
$$

That is, in the chiral case there is no renormalization of the fermionic tower (at this order) whatsoever, which is *not* what happens from the four-dimensional point of view of the previous paragraph.

7. Another example: QED_4 on a two-torus

Let us now repeat this exercise in a situation that, although probably much less interesting from the physical point of view, is much better defined as a quantum theory, namely QED_4 on a two-torus. In this example everything in the mother theory is well defined (as long as we keep shy of the Landau pole). There is no ambiguity associated as to how we define the cutoff scale in the extra-dimensional theory, or about the energy scale above which perturbation theory is not expected to be valid anymore. The reduced theory is a two-dimensional one, where all divergences are more or less trivial (essentially normal ordering). It is nevertheless possible to analyze it with the very same general techniques. We perform this computation in order to get a template to compare with our previous six-dimensional results.

7.1 The four-dimensional viewpoint

Let us then consider QED_4 on a manifold $R^2 \times S^1 \times S^1$. The action is

$$
S = \int d^4x \left(\frac{1}{4}F^2_{\mu\nu} + \bar{\psi}(\not{D} + m)\psi\right) \tag{7.1}
$$

where the abelian covariant derivative is simply:

$$
D_\mu \psi \equiv \left(\partial_\mu - eA_\mu\right)\psi \tag{7.2}
$$

The theory is renormalizable. In dimensional renormalization the counterterm is the fourth coefficient in the small-time heat kernel expansion:

$$
a_4 = \int \frac{d^4x}{(4\pi)^2}\left(\frac{2}{3}e^2\bar{F}^2_{\mu\nu} + 2e^2\bar{\eta}\gamma^\mu\bar{D}_\mu\eta + 8e^2 m\bar{\eta}\eta\right) \tag{7.3}
$$

In the cutoff theory, this is precisely the coefficient of the logaritrhmic divergence, but there is a quadratic divergence as well:

$$\Delta S = \int d^4x \left(b_2 \Lambda_{d=4}^2 + b_4 \log \frac{\Lambda_{d=4}^2}{\mu_{d=4}^2} \right) \tag{7.4}$$

where

$$a_2 = \int \frac{d^4x}{(4\pi)^2} 4m^2 \tag{7.5}$$

7.2 The two-dimensional viewpoint

In order to dimensionaly reduce the theory we consider the matrices ($a = 1, 2$)

$$\begin{aligned} \gamma_a^{(4)} &= \sigma_3 \otimes \sigma_a \\ \gamma_3^{(4)} &= \sigma_1 \otimes 1 \\ \gamma_4^{(4)} &= \sigma_2 \otimes 1 \end{aligned} \tag{7.6}$$

In that way, four-dimensional spinors split in two two-dimensional ones:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{7.7}$$

It is a simple matter to perform the integrals over the angular variables and obtain the gauge fixed action (still exact) in the two-dimensional form:

$$\begin{aligned} S = \int d^2x \sum_{n_3,n_4} \Bigg(& \bar{\psi}_n^1 \partial\!\!\!/ \psi_n^1 + \bar{\psi}_n^2 \partial\!\!\!/ \psi_n^2 + \bar{\psi}_n^1 \left(i\frac{n_3}{R_3} + \frac{n_4}{R_4} \right) \psi_n^2 - \bar{\psi}_n^2 \left(i\frac{n_3}{R_3} - \frac{n_4}{R_4} \right) \psi_n^1 + \\ & + m \left(\bar{\psi}_n^1 \psi_n^1 - \bar{\psi}_n^2 \psi_n^2 \right) - \frac{1}{2} (A_a^n)^* \left(\Box - \frac{n_3^2}{R_3^2} - \frac{n_4^2}{R_4^2} \right) A_n^a - \frac{1}{2} (A_3^n)^* \left(\Box - \frac{n_3^2}{R_3^2} - \frac{n_4^2}{R_4^2} \right) A_3^n - \\ & - \frac{1}{2} (A_4^n)^* \left(\Box - \frac{n_3^2}{R_3^2} - \frac{n_4^2}{R_4^2} \right) A_4^n - e \sum_m \left(\bar{\psi}_m^1 A\!\!\!/_{m-n} \psi_n^1 + \bar{\psi}_m^2 A\!\!\!/_{m-n} \psi_n^2 + \right. \\ & \left. + \bar{\psi}_m^1 A_3^{m-n} \psi_n^2 - \bar{\psi}_m^2 A_3^{m-n} \psi_n^1 - i \bar{\psi}_m^1 A_4^{m-n} \psi_n^2 - i \bar{\psi}_m^2 A_4^{m-n} \psi_n^1 \right) \Bigg) \end{aligned} \tag{7.8}$$

The two-dimensional coupling constant is

$$e \equiv \frac{e^{(4)}}{2\pi\sqrt{R_3 R_4}} \equiv e^{(4)} M \tag{7.9}$$

In two dimensions, gauge fields are dimensionless and so are scalar fields. Fermionic fields enjoy mass dimension 1/2. We hope that there would arise no confusion for the use of the same symbol e for both coupling constants. The zero mode of this action is

$$\begin{aligned} S = \int d^2x \Big(& \bar{\psi}^1 \partial\!\!\!/ \psi^1 + \bar{\psi}^2 \partial\!\!\!/ \psi^2 + m \left(\bar{\psi}^1 \psi^1 - \bar{\psi}^2 \psi^2 \right) - \frac{1}{2} A_a \Box A^a - \\ & - \frac{1}{2} \phi^* \Box \phi - e \left(\bar{\psi}^1 A\!\!\!/ \psi^1 + \bar{\psi}^2 A\!\!\!/ \psi^2 + \bar{\psi}^1 \phi \psi^2 - \bar{\psi}^2 \phi^* \psi^1 \right) \Big) \end{aligned} \tag{7.10}$$

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where we have represented the zero modes of all fields by the same letter without any subindex:

$$A_3^0 - iA_4^0 \equiv \phi^0 \equiv \phi \tag{7.11}$$

If we define the theory by dimensional renormalization, the counterterm associated to the above action is

$$\Delta S_{\text{zero mode}} = \frac{1}{\epsilon} a_2^{(0)} = \frac{1}{\epsilon} \int \frac{d^2x}{4\pi} 4 \left(m^2 - e^2|\phi|^2\right) \tag{7.12}$$

If instead we consider the whole tower the corresponding counterterm is given in terms of the complex mass parameter:

$$L \equiv \frac{l_4}{R_4} + i\frac{l_3}{R_3} \tag{7.13}$$

$$\Delta S_{\text{tower}} = \frac{1}{\epsilon} a_2 = \frac{1}{\epsilon} \int \frac{d^2x}{4\pi} \sum_l 4 \left(m^2 - |L|^2 - e^2 \sum_n \bar{\phi}_n^* \bar{\phi}_{-n} + e\left(L^* \bar{\phi}_0 - L\bar{\phi}_0^*\right) \right) \tag{7.14}$$

Here we have a sum of contributions from all higher modes. This is a divergent sum which needs regularization. In the expression for the tadpole, for example, we are forced to compute the sum

$$T(R) \equiv \sum_{n\in\mathbb{Z}} \frac{n}{R} \equiv \frac{1}{R} \sum_{n\in\mathbb{Z}} n \tag{7.15}$$

This can be regularized, for example, ([11]) by imposing a cutoff

$$\begin{aligned} \sum_{n=1} n &\equiv \lim_{\epsilon\to 0} \sum_{n=1} n e^{-\epsilon n} = \lim_{\epsilon\to 0} \sum_{n=1} -\frac{\partial}{\partial\epsilon} e^{-\epsilon n} = -\lim_{\epsilon\to 0} \frac{\partial}{\partial\epsilon} \sum_{n=1} e^{-\epsilon n} \\ &= -\lim_{\epsilon\to 0} \frac{\partial}{\partial\epsilon} \frac{1}{e^{\epsilon}-1} = \lim_{\epsilon\to 0} \frac{e^{\epsilon}}{(e^{\epsilon}-1)^2} = \lim_{\epsilon\to 0} \left(\frac{1}{\epsilon^2} - \frac{1}{12}\right) \end{aligned} \tag{7.16}$$

This clearly shows the divergence of the sum. When adopting a finite prescription, it is important to keep this in mind. One such finite prescription, quite natural, stems from a consideration of the laplacian operator on the extra dimensions, Δ_y, whose eigenvalues are precisely

$$\lambda_l \equiv |L|^2 \tag{7.17}$$

and the corresponding ζ function is

$$\zeta(s) \equiv \sum_{l\neq 0} \left(|L|^2\right)^{-s} \tag{7.18}$$

which happens to be a particular instance of Epstein's zeta function. This would lead to definite values for

$$\sum_l 1 \equiv \zeta(s=0) + 1 = 0 \tag{7.19}$$

and

$$\sum_l |L|^2 \equiv \zeta(-1) = 0 \tag{7.20}$$

In order to evaluate the coefficient of the tadpole, it is not possible to use this same ζ function. One possibility is to use Riemann's ζ function

$$\zeta_R(s) \equiv \sum_{n=1} n^{-s} \tag{7.21}$$

so that, for example,

$$T(R) = \frac{1}{R}(\zeta_R(-1) - \zeta_R(-1)) = 0 \tag{7.22}$$

Actually this is a unavoidable consequence of any definition in which the first of Hardy's properties of the sum of a divergent series is satisfied, namely, if $\sum a_n = S$ then $\sum \lambda a_n = \lambda S$ (cf. [12], and the discussion in [13])

It has to be acknowledged that the need to use two different zeta functions greatly diminishes the attractiveness of this whole procedure of resummation.

At any rate, in order to eliminate the tadpole, one would have in its case to shift the field:

$$\bar{\phi}_0 \to \bar{\phi}_0 - \frac{T}{2e} \tag{7.23}$$

This shift would in turn affect the fermionic masses through the Yukawa couplings and convey another contribution to the fermion mass renormalization.

When either theory is defined through a proper time cutoff, the counterterm is given precisely by

$$\Delta S = a_2 \log \frac{\Lambda^2_{d=2}}{\mu^2_{d=2}} \tag{7.24}$$

7.3 The limitations of the two-dimensional approach

Let us first concentrate upon dimensional renormalization. The mode expansion of the four-dimensional counterterm (7.3) is:

$$\begin{aligned} a_4 = \int \frac{d^2x}{(4\pi)^2} \frac{e^2}{M^2} \sum_n \Bigg(& 2\left(\bar{\eta}^1_n \not{\partial} \eta^1_n + \bar{\eta}^2_n \not{\partial} \eta^2_n + N \bar{\eta}^1_n \eta^2_n + N^* \bar{\eta}^2_n \eta^1_n\right) + \\ & + 8m\left(\bar{\eta}^1_n \eta^1_n - \bar{\eta}^2_n \eta^2_n\right) + \frac{2}{3}\Bigg(\bar{F}^{ab}_{-n} \bar{F}^n_{ab} + 2|N|^2 \bar{A}^a_{-n} \bar{A}^n_a - 4i\partial_a \bar{A}^a_{-n}\left(\frac{n_3}{R_3}\bar{A}^n_3 + \frac{n_4}{R_4}\bar{A}^n_4\right) + \\ & + 2\bar{A}_3^{-n}\left(-\Box + \frac{n_4^2}{R_4^2}\right)\bar{A}^n_3 + 2\bar{A}_4^{-n}\left(-\Box + \frac{n_3^2}{R_3^2}\right)\bar{A}^n_4 - 4\frac{n_3 n_4}{R_3 R_4}\bar{A}_3^{-n}\bar{A}^n_4\Bigg) - \\ & - 2e\sum_m \left(\bar{\eta}^1_m \bar{\not{A}}_{m-n} \eta^1_n + \bar{\eta}^2_m \bar{\not{A}}_{m-n} \eta^2_n + \bar{\eta}^1_m \bar{\phi}_{m-n} \eta^2_n - \bar{\eta}^2_m \bar{\phi}^*_{m-n} \eta^1_n\right)\Bigg) \end{aligned} \tag{7.25}$$

Which has a zero mode

$$\begin{aligned} a_4^{(0)} = \int \frac{d^2x}{(4\pi)^2} \frac{e^2}{M^2} \Bigg(& 2\left(\bar{\eta}^1 \not{\partial} \eta^1 + \bar{\eta}^2 \not{\partial} \eta^2\right) + 8m\left(\bar{\eta}^1 \eta^1 - \bar{\eta}^2 \eta^2\right) + \frac{2}{3}\bar{F}^{ab}\bar{F}_{ab} - \frac{4}{3}\bar{\phi}^* \Box \bar{\phi} - \\ & - 2e\left(\bar{\eta}^1 \bar{\not{A}} \eta^1 + \bar{\eta}^2 \bar{\not{A}} \eta^2 + \bar{\eta}^1 \bar{\phi} \eta^2 - \bar{\eta}^2 \bar{\phi}^* \eta^1\right)\Bigg) \end{aligned} \tag{7.26}$$

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In that case, it is plain that there are many differences between the detailed forms of the mode expansion of the renormalized four dimensional theory and the renormalization of the two-dimensional mode expansion of the bare four-dimensional theory.

In the cutoff theory we could be tempted to identify

$$\frac{\Lambda_{d=4}^2}{M^2} \equiv \log \frac{\Lambda_{d=2}^2}{\mu_{d=2}^2} \tag{7.27}$$

If one is willing to do this, there are two things that happen. First of all, one never recovers the two dimensional correction to the mass of the scalar field,

$$e^2|\phi|^2 \tag{7.28}$$

The reason is exactly the same as it was when reducing from six to four dimensions in our previous paper, namely, the spontaneously nature of the breaking of Lorentz symmetry of the mother theory:

$$O(1,3) \rightarrow O(1,1) \times O(2) \times O(2) \tag{7.29}$$

It is true that this correction vanishes when one considers the full tower and one is willing to regularize the sum using the zeta funcion approach. As we have pointed out, there is an implicit renormalization of the scalar mass involved in this regularization. It is nevertheless true that one can regularize the sum in such a way as to get essentially the same result for the dominant (logarithmic) divergence in both the mother and the daughter theories..

The second thing that happens, and this seems unavoidable, is that there are log log Λ^2 divergences coming from the a_4 four-dimensional counterterm, suppressed by appropiate powers of the Kaluza-Klein scale.

To conclude, even in this example, the two-dimensional theory never forgets its mother. This exercise fully supports the general conclusions of our previous reduction.

8. Conclusions

Two radically different ways to define QED_6 at a one-loop level have been explored. The lessons of this exercise seem to be as follows.

When the fundamental theory is defined in dimension higher than four using dimensional regularization, the divergences of the four dimensional theory do not match the ones of the extra-dimensional (mother) one. This is true even in the zero volume limit, when the volume of the extra dimensions is shrunk to zero, and the Kaluza-Klein scale correspondingly goes to infinity, and this is also true even when the full Kaluza-Klein tower is taken into account, as we have shown in detail in an explicit six-dimensional example.

In other words, the theory never forgets its higher dimensional origin. This is most clearly seen in the chiral limit, but appears also in the massive case, with the need of taking into account counterterms involving higher dimensional operators, whose coefficients can be computed in an unambiguous and straightforward way. We understand that a need for those counterterms has been hinted at in [2] and [3].

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The full set of four-dimensional counterterms can be easily recovered from the six-dimensional one by performing an harmonic expansion. This yields what is, in our opinion, *the* correct way of renormalizing Kaluza-Klein theories.

In the massless case (as well as when coming from an odd number of spacetime dimensions) the four-dimensional counterterms are simply not contained in the higher dimensional ones. The appropiate procedure in those cases would be, from our point of view, to compute in the mother theory (in which finite results are obtained through the use of dimensional regularization), and then perform the mode expansion.

Alternatively, when the quantum theory is defined through a proper time cutoff, we recover the four dimensional logarithmic divergences via a tuning of the six-dimensional cutoff. There are then calculable log log Λ^2 divergences coming from the six-dimensional logarithmic divergences as a reminder of the sicknesses of the mother theory. Those divergences are, however, suppressed by appropiate powers of the compactification scale, which means that they are multiplied by a small coefficient at energies at which six-dimensional perturbation theory is reliable (essentially $E/M \ll \alpha_{d=4}^{-1}$).

In neither case do we find from six dimensions corrections to the potential energy of the four-dimensional singlet scalars associated to the zero modes of the extra-dimensional legs of the gauge field. This being true for the zero mode, is clearly a low energy effect, well within the range of validity of the one-loop six-dimensional calculation . Those corrections are found in four dimensions because there is no gauge symmetry to prevent that to happen.

We have repeated the analysis for QED_4 on a two-torus, getting similar results. This is very important, because there is now no ambiguity as to how to define the extra-dimensional theory. This shows, in our opinion, that our main results do not stem from the ambiguities inherent in any practical approach to a non-renormalizable theory.

There are no special difficulties with either odd-dimensional spaces (cf. for example [14]) or massless fermions from the viewpoint of the cutoff theory. Let us finally stress that the strictest equivalence *does work* for free theories coupled to the gravitational field, so that all the effects we have studied here are due to the interaction.

Our results have obvious applications to the study of the range of validity of the low energy effective four-dimensional models when studying Kaluza-Klein theories (cf. for example [15]) because our framework is consistent by construction (that is, to the extent that the six-dimensional model is consistent).

Although a very simple abelian model has been studied in this paper as an example, we do not expect our main results to change in more complicated (non abelian) situations. Besides obvious extensions, like supersymmetry and chiral fermions, it would be interesting to study the effects of a nontrivial gravitational field, as well as the physics of codimension one terms in the action (like the presence of branes in it).

A most interesting related issue is how the full mother theory compares with the ultraviolet completion as implied by deconstruction (cf. for example [16]).

Work is in progress in several of these matters.

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A. The one-loop effective action as a superdeterminant

In order to get acquainted with the heat kernel techniques let us repeat a well-known computation , namely, the divergent part of the effective action of quantum electrodynamics in $d = 4$ dimensions. In doing so, we shall employ a technique first introduced by I. Jack and H. Osborn [17] (cf. also [18]), which is exceedingly convenient in case there are non-vanishing fermionic background fields (and which is extensively used in the main text). The main idea will be to represent the combination of fermionic and bosonic determinants as a single superdeterminant, or berezinian, as is sometimes refered to.

The euclidean action reads

$$\mathcal{L} = \bar{\chi}\gamma^M \partial_M \chi - e\bar{\chi}\gamma^M A_M \chi + m\bar{\chi}\chi + \frac{1}{4} F_{MN} F^{MN} \tag{A.1}$$

We now split the fields in a classical a a quantum part:

$$\begin{aligned} A_M &= \bar{A}_M + \phi_M \\ \chi &= \eta + \psi \end{aligned} \tag{A.2}$$

where the backgrounds do obey the classical equations of motion, i.e.,

$$\begin{aligned} \left(\gamma^M (\partial_M - e\bar{A}_M) + m\right) \eta &= 0 \\ \partial_M \bar{F}^{MN} + e\bar{\eta}\gamma^N \eta &= 0 \end{aligned} \tag{A.3}$$

Keeping only the terms quadratic in the quantum fields and sticking to Feynman's gauge leads to:

$$\mathcal{L} = \bar{\psi}\gamma^M \partial_M \psi - e\bar{\psi}\gamma^M \bar{A}_M \psi + m\bar{\psi}\psi - e\bar{\eta}\gamma^M \phi_M \psi - e\bar{\psi}\gamma^M \phi_M \eta - \frac{1}{2}\phi_M \partial_N \partial^N \phi^M \tag{A.4}$$

which can be written as:

$$\mathcal{L} = \frac{1}{2}\phi_M A_{MN} \phi_N + \bar{\psi} B \psi + \phi_N \bar{\Gamma}_N \psi + \bar{\psi} \Gamma_M \phi_M \tag{A.5}$$

with

$$\begin{aligned} A &= -\partial_R \partial^R \delta_{MN} \\ B &= \gamma^M \partial_M - e\gamma^M \bar{A}_M + m \\ \Gamma_N &= -e\gamma_N \eta \\ \bar{\Gamma}_M &= -e\bar{\eta}\gamma_M \end{aligned} \tag{A.6}$$

This can equally well be expressed (cf. [18]) in terms of the supermatrix

$$\Delta = \begin{pmatrix} A_{MN} & \sqrt{\frac{2}{\mu}}\bar{\Gamma}_M \gamma_5 B \gamma_5 \\ \sqrt{2\mu}\Gamma_N & B\gamma_5 B\gamma_5 \end{pmatrix} \tag{A.7}$$

as

$$S = \int d^4x \bar{\xi} \Delta \xi \tag{A.8}$$

with $\xi = (\phi_M, \psi)$.

Our main interest is the computation of the free energy,W:

$$Z \equiv e^{-W} \equiv e^{-\bar{S}} \int \mathcal{D}\xi e^{-S[\xi]} \tag{A.9}$$

The free energy

$$W \equiv \log\, Z \tag{A.10}$$

is then given by

$$W \equiv \bar{S} + \frac{1}{2} \log\, \mathrm{sdet}\, \Delta \tag{A.11}$$

The superdeterminant, or berezinian of a supermatrix M involving bosonic $(+)$ and fermionic $(-)$ entries

$$M = \begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{pmatrix} \tag{A.12}$$

is defined by

$$\mathrm{ber}\, M \equiv \mathrm{sdet}\, M \equiv \det M_{++} \det{}^{-1} \left(M_{--} - M_{-+} M_{++}^{-1} M_{+-} \right) \tag{A.13}$$

We have introduced an arbitrary mass scale μ for dimensional reasons. In the present situation,

$$\Delta = \begin{pmatrix} -\partial_R \partial^R \delta_{MN} & \sqrt{\frac{2}{\mu}} e \left(\bar{\eta} \gamma^M \gamma^R \bar{D}_R - m \bar{\eta} \gamma^M \right) \\ -\sqrt{2\mu} e \gamma_N \eta & -\bar{D}_M \bar{D}^M + \frac{e}{2} \gamma^M \gamma^N \bar{F}_{MN} + m^2 \end{pmatrix} \tag{A.14}$$

This supermatrix operator enjoys a Laplacian form $\Delta = -D_M D^M + Y$ with $D_M = \partial_M + X_M$ and the supermatrices

$$X_M = \begin{pmatrix} 0 & \frac{-e}{\sqrt{2\mu}} \bar{\eta} \gamma^R \gamma_M \\ 0 & -e \bar{A}_M \end{pmatrix} \tag{A.15}$$

And

$$Y = \begin{pmatrix} 0 & \frac{-e}{\sqrt{2\mu}} \left(2m \bar{\eta} \gamma^M + \bar{D}_R \bar{\eta} \gamma^M \gamma^R \right) \\ -\sqrt{2\mu} e \gamma_N \eta & \frac{e}{2} \gamma^M \gamma^N \bar{F}_{MN} + m^2 \end{pmatrix} \tag{A.16}$$

Once we have reduced our problem to the computation of the determinant of a supermatrix the divergent part of the effective action is given by the $a_4(\Delta)$ coefficient in the heat kernel expansion

$$a_4(\Delta) = \int \frac{d^d x}{(4\pi)^{d/2}}\, str \left(\frac{1}{2} Y^2 + \frac{1}{12} X_{MN}^2 \right) \tag{A.17}$$

Where as usual X_{MN} is the field strength asociated with X_M and str denotes super trace. In our case the field strength supermatrix is

$$X_{MN} = \begin{pmatrix} 0 & \frac{-e}{\sqrt{2\mu}} \left(\bar{D}_M \bar{\eta} \gamma^R \gamma_N - \bar{D}_N \bar{\eta} \gamma^R \gamma_M \right) \\ 0 & -e \bar{F}_{MN} \end{pmatrix} \tag{A.18}$$

Which after squaring and tracing gives a contribution

$$\frac{1}{12} str\, X_{MN}^2 = -\frac{2^{[d/2]}}{12} e^2 \bar{F}_{MN}^2 \tag{A.19}$$

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While the contribution from Y^2 is

$$\frac{1}{2}str\,Y^2 = e^2(d-2)\bar{\eta}\gamma^M\partial_M\eta - e^3(d-2)\bar{\eta}\gamma^M\bar{A}_M\eta + 2de^2m\bar{\eta}\eta + \frac{2^{[d/2]}}{4}e^2\bar{F}^2_{MN} \quad \text{(A.20)}$$

Finally we can write the coefficient in four dimensions

$$a_4(\Delta) = \int \frac{d^4x}{(4\pi)^2}\left(\frac{2}{3}e^2\bar{F}^2_{MN} + 2e^2\bar{\eta}\gamma^M\bar{D}_M\eta + 8e^2m\bar{\eta}\eta\right) \quad \text{(A.21)}$$

which coincides with the result obtained through the application of the classical 't Hooft algorithm [19, 20].

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